Magnetic Response in Quantized Spin Hall Phase of Correlated Electrons

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We investigate the magnetic response in the quantized spin Hall (SH) phase of a layered-honeycomb lattice system with intrinsic spin-orbit coupling $\lambda_{\rm SO}$ and on-site Hubbard U. The response is characterized by the parameter $g = 4Ua^2d/3$, where a and d are the lattice constant and interlayer distance, respectively. When $g < (\sigma_{xy}^{s2}\mu)^{-1}$, where σ_{xy}^s is the quantized spin Hall conductivity and μ is the magnetic permeability, the magnetic field inside the sample oscillates spatially. The oscillation vanishes in the non-interacting limit $U \to 0$. When $g > (\sigma_{xy}^{s2}\mu)^{-1}$, the system shows perfect diamagnetism, i.e., the Meissner effect occurs. We find that a superlattice structure with large a is favorable for observing these phenomena. We also point out that, as a result of Zeeman coupling, the topologically protected helical edge states show weak diamagnetism that is independent of g.

KEYWORDS: magnetic response, quantized spin Hall effect, Kane-Mele model, intrinsic spin-orbit interaction, electron correlation, layered-honeycomb structure, superlattice, topological BF term, superconductivity

1. Introduction

The aim of this paper is to discuss the magnetic response of the quantized spin Hall (SH) phase of correlated electrons on the layered-honeycomb lattice.

It has been shown that the quantum SH effect occurs in a non-interacting electron system with intrinsic spin-orbit coupling $\lambda_{\rm SO}$, like the Kane-Mele (KM) model.[1–3] The investigation of the quantum SH system with on-site Hubbard U is now one of the current topics,[4–6] and the phase diagram has been obtained.[5,6] In ref. 7, magnetic response in the quantized SH phase of the layered KM model with U (spin-conserving limit of the topological band insulating phase in ref. 6) was discussed. Such a model can be applied to correlated electrons in the system with honeycomb layers, such as some Ir-based oxides.[4] The correlation is characterized by the parameter $g \propto Ua^2d$ in the low-energy long-wavelength regime (a; lattice constant, d; interlayer distance), and the London equation for the Meissner effect has been obtained in the large g limit.[7] The discussion on general g has not yet been given. The role of the topologically protected helical edge states, which is the hallmark of the quantum SH effect,[1,2,8–10] has also not been taken into account.

In this study, we clarify the magnetic response for general g in the quantized SH phase. First, we take into account contributions from bulk states and later consider those from the helical edge states. When $g < (\sigma_{xy}^{s2}\mu)^{-1}$, where σ_{xy}^s and μ are the quantized spin Hall conductivity (SHC) and magnetic permeability, respectively, the magnetic field inside the sample oscillates spatially around a constant value. The oscillation vanishes in the non-interacting limit $U \to 0$. When $g > (\sigma_{xy}^{s2}\mu)^{-1}$, the general solution for the magnetic field becomes a superposition of the homogeneous and damping parts. We find that the damping part is energetically favored, thus, the Meissner effect occurs. Then, we consider the contribution from the helical edge state. As a result of Zeeman coupling, it is shown that the state exhibits weak diamagnetism that is independent of g.

The argument we will present is focused on the quantized SH phase. Here, we mention that we have an upper limit for $g(\propto Ua^2d)$. It has been pointed out that the system shows a phase transition from the quantized SH phase to the topological Mott insulating phase when U becomes larger.[5,6,11] The lattice constant a also should not be too large, since we consider the long-wavelength effective theory.

This paper is organized as follows. In § 2, we introduce the layered KM model with on-site Hubbard U. In § 3, we integrate out Fermion and obtain a one-loop effective Lagrangian in the quantized SH phase. In § 4, we discuss the magnetic response. In § 5, we estimate the contribution from the helical edge state. In § 6, we comment on the relation to the superconductivity and our system. We use $\hbar = c = 1$ unit and the Minkovskian metric $g^{\mu\nu} = diag(1, -1, -1)$, where $\mu, \nu = 0, x, y$. Summations run over repeated Greek indices.

2. Layered KM Model with an Electron Correlation

We consider electrons on the layered honeycomb lattice. We assume that interlayer coupling is negligibly small and each layer is described by the KM model.[6,7] One of the essential ingredients of the KM model[1,2] is the intrinsic SO coupling $\lambda_{\rm SO}$. We can say, not strictly but intuitively, that $\lambda_{\rm SO}$ gives an effective magnetic field depending on spin. It also gives an electronic excitation gap,[2]

$$\Delta = 3\sqrt{3}\lambda_{\rm SO}. \tag{1}$$

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Thus, as an analog of the quantum Hall effect, we see quantization of the SHC,[1,2)]

$$\sigma^s_{xy} = \frac{e}{2\pi d}\frac{\Delta}{|\Delta|}, \tag{2}$$

where d is the interlayer distance. The model can also have the Rashba extrinsic SO coupling λ_R, which breaks the inversion symmetry and is induced by an electric field perpendicular to the honeycomb lattice plane. The term also breaks the conservation of electron spin S_z. Hereafter, we consider the case of $\lambda_R = 0$.

We add the on-site Coulomb repulsion $U > 0$. The Hamiltonian per layer is

$$\begin{aligned} H &= t\sum_{<ij>} c_i^\dagger c_j + i\lambda_{\rm SO}\sum_{<<ij>>}\nu_{ij}c_i^\dagger s_z c_j \\ &\quad + U\sum_i n_{i\uparrow}n_{i\downarrow}, \end{aligned} \tag{3}$$

where c_i ($c_i^\dagger$) is the annihilation (creation) operator of an electron with spin at the i-th site and t is the nearest neighbor hopping. The second term is the intrinsic SO term consisting of the next nearest neighbor hopping, and $\nu_{ij} = \frac{2}{\sqrt{3}}(\hat{\boldsymbol{d}}_1\times\hat{\boldsymbol{d}}_2)_z = \pm 1$, where $\hat{\boldsymbol{d}}_1$ and $\hat{\boldsymbol{d}}_2$ are unit vectors along the two bonds where the electron moving from site j to i passes.

Let us discuss how to deal with the electron correlation U. On-site Coulomb repulsion can be written by the on-site spin-spin interaction

$$Un_{i\uparrow}n_{i\downarrow} = \frac{U}{2}(n_{i\uparrow}+n_{i\downarrow}) - \frac{U}{6}(c_i^\dagger\vec{s}c_i)^2. \tag{4}$$

The first term in the *r.h.s* merely gives the renormalization for the chemical potential and can be neglected. We introduce the auxiliary field $\vec{\varphi}_i$, which is a three-component vector in the spin space, and use the Stratonovich-Hubbard transformation,[12)] $H \to H_{SH} = H + \Delta H$, where

$$\begin{aligned} \Delta H &= \frac{U}{6}\sum_i (c_i^\dagger\vec{s}c_i - \frac{3}{2U}\vec{\varphi}_i)^2, &(5)\\ H_{SH} &= t\sum_{<ij>} c_i^\dagger c_j + i\lambda_{\rm SO}\sum_{<<ij>>}\nu_{ij}c_i^\dagger s_z c_j \\ &\quad - \sum_i \vec{\varphi}_i\cdot c_i^\dagger\frac{\vec{s}}{2}c_i + \frac{3}{8U}\sum_i|\vec{\varphi}_i|^2. &(6) \end{aligned}$$

The spin-spin interaction is eliminated in appearance, but instead, we have coupling between $\vec{\varphi}_i$ and the electron spin, and a quadratic term of $\vec{\varphi}_i$.

We consider the continuum limit and take into account the low-energy electronic excitations around K and K' points in the Brillouin Zone,[1,2)] i.e., we omit the inter valley scattering. We introduce the electromagnetic $U(1)$ gauge field A_μ and $SU(2)$ spin gauge field $\vec{a}_\mu$ via the covariant derivative

$$iD_\mu = i\partial_\mu - eA_\mu + \vec{a}_\mu\cdot\frac{\vec{s}}{2}, \tag{7}$$

where $\vec{a}_0 = \vec{\varphi}$ (the auxiliary field in the continuum limit) and $\vec{\boldsymbol{a}}$ is an external field introduced artificially to estimate the spin current. We define a parameter

$$g = \frac{4Ua^2d}{3}, \tag{8}$$

where a is the lattice constant, and the microscopic Lagrangian density is[13)]

$$\begin{aligned} \mathcal{L} &= \Psi^\dagger\left\{iD_0 - iv(D_x\tau_z\sigma_x + D_y\sigma_y) + \Delta\tau_z\sigma_z s_z\right\}\Psi \\ &\quad + \frac{\epsilon_0 E^2}{2} - \frac{B^2}{2\mu_0} - \frac{1}{2g}|\vec{a_0}|^2, \end{aligned} \tag{9}$$

where $\Psi = \Psi_{\tau\sigma s}$ is the eight-component fermion field labeled by the eigenvalues of the diagonal components of valley spin $\vec{\tau}$, sublattice spin $\vec{\sigma}$, and real spin $\vec{s}/2$. The parameter v is the Fermi velocity when the system is in the metallic state, and ϵ_0 and μ_0 denote the vacuum values of the dielectric constant and magnetic permeability, respectively. Note that, except for the last term, the Lagrangian (9) possesses the $U(1)_{\rm em}\times U(1)_z$ local gauge symmetry. The $SU(2)$ gauge symmetry is broken down to $U(1)_z$, since the SO term contains s_z.

3. One-Loop Effective Lagrangian in the Quantized SH Phase

The phase diagram for correlated electrons on pyroclore and honeycomb lattices with $\lambda_{\rm SO}$ has been discussed using the slave-rotor model.[5,6)] The system with strong $\lambda_{\rm SO}$ and small U is in the quantized SH phase when the spin conservation is preserved. As U increases, the band gap closes and the Mott gap opens instead, i.e., band-Mott transition occurs. We focus on the quantized SH phase.

The derivation of the effective action in this phase is equivalent to that presented in ref. 14, although the physical meaning of the spin gauge field is different. We integrate out Ψ from eq. (9) and obtain the one-loop effective Lagrangian for the gauge fields. We assume that the amplitude of the gauge fields is small and use the Gaussian approximation. We consider fields with a length scale (temporal scale) of modulation that is sufficiently gradual compared with a (Δ), and use the long-wavelength (low-frequency) approximation.

The result is[14)]

$$\begin{aligned} \mathcal{L}_{\rm eff} &= -\frac{1}{2g}a_0^{z2} + \mathcal{L}_{\rm ind}, &(10)\\ \mathcal{L}_{\rm ind} &= \sigma^s_{xy}\epsilon^{\mu\rho\nu}a^z_\mu\partial_\rho A_\nu + \frac{\epsilon E^2}{2} - \frac{B^2}{2\mu} + \frac{\epsilon_s e_s^2}{2} - \frac{b_s^2}{2\mu_s} \\ &\quad + (\text{terms independent of } a^z_\mu \text{ and } A_\mu), &(11) \end{aligned}$$

where $\mathcal{L}_{\rm ind}$ stands for the induced part of the effective Lagrangian, $\epsilon^{012} = \epsilon^{120} = \epsilon^{201} = -\epsilon^{021} = -\epsilon^{210} = -\epsilon^{102} = 1$, and $\boldsymbol{e}_s = -\dot{\boldsymbol{a}}^z - \boldsymbol{\nabla}a_0^z$ and $b_s = \sum_{ij}\epsilon^{ij}\partial_i a^z_j$ with $\epsilon^{12} = -\epsilon^{21} = 1$ are the spin electric field and spin magnetic field, respectively. The first term in eq. (11) is the topological BF term,[7,14–18)] which plays an important role in our discussion. The coefficient is the quantized SHC given in eq. (2). Note that only a^z_μ couples to the electromagnetic gauge fields. This comes from the fact that the $SU(2)$ symmetry is broken down to $U(1)_z$ symmetry by the SO coupling. We comment on the more

detailed properties of this term in the Appendix. The Maxwell term is renormalized as[14, 19)]

$$\epsilon = \epsilon_0 + \delta\epsilon, \tag{12}$$

$$\frac{1}{\mu} = \frac{1}{\mu_0} + \frac{1}{\delta\mu}, \tag{13}$$

$$\delta\epsilon \equiv \frac{e^2}{6\pi|\Delta|d}, \tag{14}$$

$$\frac{1}{\delta\mu} \equiv \frac{e^2 v^2}{6\pi|\Delta|d}, \tag{15}$$

$$\epsilon_s \equiv \frac{\delta\epsilon}{4e^2}, \tag{16}$$

$$\frac{1}{\mu_s} \equiv \frac{1}{4e^2\delta\mu}. \tag{17}$$

By using the relations $\frac{e^2}{4\pi\epsilon_0} \simeq 1/137$ and $\epsilon_0\mu_0 = 1$ and also the parameters in Table. I, which are relevant for the honeycomb-layered insulator Na_2IrO_3,[4)] we obtain $\epsilon_0/\frac{e^2}{6\pi|\Delta|d} = 0.5$ and $\mu_0 \cdot \frac{e^2v^2}{6\pi|\Delta|d} = 2\times 10^{-8}$, i.e., $\mu \simeq \mu_0$. The elastic term for $a_0^z(=\varphi^z)$ is also induced. We can recognize that any potential terms (i.e., zeroth-order terms with respect to the derivative ∂_μ) of A_μ and also a_μ^z in $\mathcal{L}_{\rm ind}$ are absent because of the presence of $U(1)_{\rm em}\times U(1)_z$ gauge symmetry in the fermionic part of the microscopic Lagrangian (9). Thus, the low-energy and long-wavelength physics of A_μ and a_μ^z is described definitely by eq. (10).

Table I. Parameters used for estimations. These are typical values for Na_2IrO_3,[4)] which is a honeycomb-layered insulator with $\lambda_{\rm SO}$ and electron correlation.

Δ	U	d	a	v
0.5eV	0.5eV	10Å	10Å	3×10^4m/s

4. Magnetic Response

We consider the static magnetic response. Here, we set $\vec{\mathbf{a}} = const$. The equations of motion for spin chemical potential a_0^z (originally, this is the auxiliary field φ^z) and magnetic field B obtained from eq. (10) are

$$\epsilon_s\nabla^2 a_0^z + \frac{1}{g}a_0^z = \sigma_{xy}^s B, \tag{18}$$

$$\frac{1}{\mu}\sum_{j=x,y}\epsilon_{ij}\nabla_j B = \sigma_{xy}^s\sum_{j=x,y}\epsilon_{ij}\nabla_j a_0^z. \tag{19}$$

The *r.h.s.* of eqs. (18) and (19) are the results from the BF term [see eqs. (A·3) and (A·4) in the Appendix], respectively.

We consider a sample in $x \geq 0$, and apply the homogeneous magnetic field B_0 parallel to z-axis, which is perpendicular to the layers. Around the boundary $x = 0$, we have the helical edge mode.[1, 2)] The contribution from the edge mode is discussed in the next section. It will be shown that the edge mode gives a small correction via Zeeman coupling.

Obviously, the fields depend on x only, and the general solution is

$$\begin{pmatrix} a_0^z(x) \\ B(x)\end{pmatrix} = \alpha\begin{pmatrix} g\sigma_{xy}^s \\ 1\end{pmatrix} + \beta_+\begin{pmatrix} 1/\sigma_{xy}^s\mu \\ 1\end{pmatrix}e^{ik_0x} + \beta_-\begin{pmatrix} 1/\sigma_{xy}^s\mu \\ 1\end{pmatrix}e^{-ik_0x}, \tag{20}$$

$$k_0 \equiv \mathcal{C}\sqrt{\frac{1-s}{s}}, \tag{21}$$

$$\mathcal{C} \equiv \sigma_{xy}^s\sqrt{\frac{\mu}{\epsilon_s}}, \tag{22}$$

$$s \equiv \sigma_{xy}^{s2}\mu g \geq 0, \tag{23}$$

where α, and $\beta_\pm$ are arbitrary constants. The energy functional of the fields is

$$Ene. = \int d^3x\left\{\frac{1}{2\mu}B^2 + \frac{\sigma_{xy}^2\mu}{2\mathcal{C}}(\boldsymbol{\nabla}a_0^z)^2 + \frac{1}{2g}a_0^2\right\}, \tag{24}$$

which will be used to determine the constants. Note that the BF term is absent, since the term does not consume energy.[13)]

4.1 Oscillation effect

For $s = \sigma_{xy}^{s2}\mu g < 1$, k_0 is real. The Dirichlet-type boundary condition for $B(x)$ at $x = 0$ is

$$\frac{B(0)}{\mu} = \frac{B_0}{\mu_0}. \tag{25}$$

We also assume that the fields are real and that the energy functional (24) per period $2\pi/k_0$ along the x-direction takes a minimum value. Then, we obtain

$$B(x) = B_0\frac{\mu}{\mu_0}\left\{\frac{1}{1+X} + \frac{X}{1+X}\cos k_0x\right\}, \tag{26}$$

$$a_0^z(x) = B_0\frac{\mu}{\mu_0}\left\{\frac{g\sigma_{xy}^s}{1+X} + \frac{1}{\sigma_{xy}^s\mu}\frac{X}{1+X}\cos k_0x\right\}, \tag{27}$$

$$X = \frac{2s^2(1+2s)}{1+s^2}. \tag{28}$$

The first term in the *r.h.s.* of eq. (26) is homogeneous, and the second one shows oscillation of the magnetic field. It is reasonable that the amplitude of oscillation vanishes in the non-interacting limit $U \to 0$ [see eqs. (8), (23), and (28)].

Using the parameters shown in Table I, we obtain $s \simeq 7.0\times 10^{-6}$, and it is hard to observe the oscillation. If we have the lattice with $a = 350$ nm, we obtain $s \simeq 0.99$. In this case, the amplitude of oscillation is about 75 % of $B_0\mu/\mu_0$. The wavelength of the oscillation is

$$\lambda_{\rm osci.} = \frac{2\pi}{\mathcal{C}}\sqrt{\frac{s}{1-s}} \simeq 3\mu\text{m}. \tag{29}$$

This result is consistent with a long-wavelength approximation since $\lambda_{\rm osci.} \gg a$.

4.2 Meissner effect

For $s>1$, k_0 becomes pure imaginary. We introduce a real value,

$$\kappa_0 \equiv -ik_0 = \mathcal{C}\sqrt{\frac{s-1}{s}}. \tag{30}$$

We impose solution (20) to be real and finite, and use boundary condition (25). The solution that gives the minimum of energy functional (24) is

$$B(x) = B_0\frac{\mu}{\mu_0}e^{-\kappa_0 x}, \tag{31}$$

$$a_0^z(x) = \frac{B_0}{\sigma_{xy}^s\mu_0}e^{-\kappa_0 x}. \tag{32}$$

It is obvious that the energy of these solutions converges because of the exponential damping. The homogeneous part should not appear, since energy functional (24) diverges. The above solutions remind us of the Meissner effect with the penetration depth

$$\lambda_{\text{pen.}} = \frac{2\pi}{\kappa_0} = \frac{2\pi}{\mathcal{C}}\sqrt{\frac{s}{s-1}}. \tag{33}$$

It has been pointed out that eqs. (18) and (19) lead to the London equation for $s \gg 1$.[7] In this limit, we can neglect the $1/g$ term in the *l.h.s.* of eq. (18) and obtain the London equation by taking the rotation of both sides of eq. (19) and using eq. (18). Amazingly, we obtain the Meissner effect without the London equation in this paper. Namely, the condition for the Meissner effect is weakened as $s>1$, instead of $s\gg 1$.

We can see from eq. (33), large s strengthens the screening. On the other hand, U should not be too large, since we are discussing the quantized SH (topological band insulating) phase.[5,6] Thus, we can see, from eqs. (2), (8), and (23), that large a and small d are favorable for observing the Meissner effect. We also note that large $\Delta \propto \lambda_{\text{SO}}$ shortens the penetration depth, since $\mathcal{C}\propto\Delta^{1/2}$ [see eqs. (14),(22), and (33)].

The physical picture of this Meissner effect is considered to be as follows: the spin-orbit coupling opens the topological gap and compensates the energy loss coming from the screening of the magnetic field.

If we have a lattice with $a = 380$ nm, instead of $a = 10$ Å in Table I, we obtain $s \simeq 1.012$ and the penetration depth of the magnetic field is estimated to be $\lambda_{\text{pen.}} \simeq 2.8\ \mu$ m. This result is consistent with long-wavelength approximation, since $\lambda_{\text{pen.}} \gg a$.

4.3 Quantized current in topologically non trivial insulators

The existence of charged current is indicated, since the magnetic field is spatially dependent. It may sound curious that current flows in an insulating system. We note that the current is a result of a combination of spin accumulation (A·3) and the dual quantized SH effect (A·4) derived from the BF term [see Appendix]. The spin accumulation causes the spin chemical potential a_0^z, and the quantized electric current flows perpendicular to $-\boldsymbol{\nabla}a_0^z$, i.e., the spin electric field. Therefore, the origin of the current has a direct analog of the quantized Hall effect,[20] in which non dissipative quantized transport carried *not* by the excited state but by the ground state occurs. These are typical transport phenomena in band insulators with non trivial topology.[20,21] We note that such a non dissipative charge transport *does not* need the spontaneous $U(1)_{\text{em}}$ symmetry breaking.

In this section, we have considered the bulk state only and the helical edge state was not taken into account. We will discuss, in the next section, that, as a result of the Zeeman effect, the edge state shows weak diamagnetism independent of the correlation parameter g, and gives a correction to the results of eqs. (26), (27), (31), and (32).

5. Contribution from the Helical Edge State

In this section, we consider the edge in more detail. It is important to estimate the contribution from the helical edge mode, which is the hallmark of the quantum SH system.[1,2,8–10] We assume that the edge modes are completely localized at the boundary and neglect the broadness. First, we neglect Zeeman coupling, and later, we take it into account.

5.1 Without Zeeman coupling

In this subsection, we omit Zeeman coupling and show that the helical edge mode has a non-gauge-invariant contribution to the magnetic response. This contribution is cancelled out by the anomalous boundary response as a result of the bulk BF term. Namely, only the bulk contribution taken into account in the previous discussion is relevant. This anomaly cancellation, renowned as the bulk-edge correspondence, is a natural consequence of the gauge invariance, and exactly the reason for the presence of the helical edge mode.[15–17,22–26]

We consider the layered system in $x \geq 0$, and each layer is normal to the z-axis. The helical edge mode can be modeled by a pair of quasi one-dimensional (1D) massless Dirac fermions with opposite spin and velocity. For simplicity, we assume that edge fermions are localized completely at the boundary. We also assume that the many-body interaction between edge fermions is irrelevant, since we are discussing the integral quantized phase. We introduce gauge couplings for the fermions in a covariant manner, and integrate out the fermions. This calculation is a direct analog of that for the chiral edge mode in the quantized Hall effect, and can be modeled by a single 1D massless Dirac fermion.[24] Then, we obtain the response to the electromagnetic gauge field

$$j_\alpha^{(edge)z} = -\delta(x)\frac{\sigma_{xy}^s}{2}\times \tag{34}$$
$$\left\{g_{\alpha\beta} - (g_{\alpha\alpha'}+\epsilon_{\alpha\alpha'})\frac{\partial^{\alpha'}\partial^{\beta'}}{\partial^2}(g_{\beta'\beta}-\epsilon_{\beta'\beta})\right\}A^\beta,$$

where $\alpha,\beta = 0, y$ denotes the space-time coordinate at the quasi-1D edge lying in the $x=0$ plane, $g^{\alpha\beta} = diag(v^{-2},-1)$, $\partial^2 = g_{\alpha\beta}\partial^\alpha\partial^\beta = (\partial_0/v)^2 - \partial_y^2$, and $\epsilon_{\alpha\beta} = -\epsilon_{\beta\alpha}$. This current is anomalous since it is not invariant under the gauge transformation $A_\alpha \to A_\alpha + \partial_\alpha\xi$, where ξ is a regular function.

In the bulk, we have the BF term. The term in the

symmetrized form can be written as

$$\mathcal{L}_{BF} = \frac{\sigma^s_{xy}}{2}\theta(x)\epsilon^{\mu\rho\nu}(a^z_\mu\partial_\rho A_\nu + A_\mu\partial_\rho a^z_\nu). \quad (35)$$

We see that the response to the external field A_μ has a boundary term [see the last term]:

$$\begin{aligned} j^{(BF)z}_\mu &= \frac{\partial\mathcal{L}_{BF}}{\partial a^{z\mu}} \quad (36)\\ &= \theta(x)\sigma^s_{xy}\epsilon_{\mu\rho\nu}\partial^\rho A^\nu - \delta(x)\frac{\sigma^s_{xy}}{2}\epsilon_{x\mu\rho}A^\rho. \end{aligned}$$

Note that this is spin density ($\mu = 0$) or spin current density ($\mu = i$), and the boundary term breaks the gauge invariance.

We consider the response to the static magnetic field. Around the edge $x = 0$, we may write $A_0 = A_x = 0$ and $A_y = B(0)x$. Thus, $A_\alpha = \partial_\alpha \{B(0)xy\}$ and from eq. (34), we obtain

$$j^{(\mathrm{edge})z}_0 = \delta(x)\frac{\sigma^s_{xy}}{2}A_y, \quad (37)$$

$$j^{(\mathrm{edge})z}_y = 0, \quad (38)$$

and from eq. (36),

$$j^{(\mathrm{BF})z}_0 = \theta(x)\sigma^z_{xy}B - \delta(x)\frac{\sigma^s_{xy}}{2}A_y, \quad (39)$$

$$j^{(\mathrm{BF})z}_y = 0. \quad (40)$$

Therefore, the non-gauge-invariant boundary terms cancell each other out. Total spin density, which should be inserted in the *r.h.s.* of eq. (18), is

$$\rho_s = j^{(\mathrm{BF})z}_0 + j^{(\mathrm{edge})z}_0 = \theta(x)\sigma^s_{xy}B. \quad (41)$$

Thus, we conclude that only the bulk contribution is relevant to the magnetic response when we neglect Zeeman coupling.

5.2 With Zeeman coupling

We can see that the helical edge mode contributes to the magnetic response as a result of Zeeman coupling.[27] We assume that the Zeeman splitting energy is much less than the bulk band gap Δ. As mentioned, the helical edge mode is described by a pair of quasi-1D gapless Dirac fermions. When SHC is given by eq. (2), i.e., quantized as $+1$ with the unit of $e/2\pi$, the edge spectrum is

$$E^{\uparrow,\downarrow}_{k_y} = \mp vk_y, \quad (42)$$

where k_y is the momentum measured from the Fermi points, and $-v$ and $+v$ ($v > 0$) are velocities for up-spin and down-spin fermions, respectively. In the case of SHC quantized as -1, the signs of velocities are opposite to each other.

As usual metals, Fermi points for up- and down-spin fermions are split by the Zeeman effect, and the number density for each spin is generated as

$$\begin{aligned} \Delta n^{\uparrow,\downarrow} &= \pm\left\{\left(\frac{\mu_B B(0)}{v}\right)/\left(\frac{2\pi}{L}\right)\right\}\frac{\delta(x)}{Ld}\\ &= \pm\frac{\mu_B B(0)}{2\pi vd}\delta(x), \quad (43) \end{aligned}$$

where μ_B is the Bohr magneton, and L ($\gg a$) is the total length of the sample edge. Induced spin density is

$$\begin{aligned} \Delta\rho_s &= \frac{1}{2}(\Delta n^\uparrow - \Delta n^\downarrow)\\ &= \frac{\mu_B B(0)}{2\pi vd}\delta(x). \quad (44) \end{aligned}$$

Because of the characteristic feature of the spectrum (42), we have induced current density. For each spin,

$$\Delta j^{\uparrow,\downarrow} = \pm ev\Delta n^{\uparrow,\downarrow}, \quad (45)$$

and in total, we have

$$\begin{aligned} \Delta j &= \Delta j^\uparrow + \Delta j^\downarrow\\ &= \frac{e\mu_B}{\pi d}B(0)\delta(x). \quad (46) \end{aligned}$$

Therefore, the equations of motion (18) and (19) are modified as

$$\epsilon_s\frac{d^2a^z_0}{dx^2} + \frac{a^z_0}{g} = \sigma^s_{xy}B + \frac{\mu_B B(0)}{2\pi v}\delta(x), \quad (47)$$

$$\frac{1}{\mu}\frac{dB}{dx} = \sigma^s_{xy}\frac{da^z_0}{dx} - \frac{e\mu_B B(0)}{\pi d}\delta(x). \quad (48)$$

Instead of eq. (25), the Dirichlet boundary condition for B at $x = 0$ is

$$\frac{B(0)}{\mu} = \frac{B_0}{\mu_0} - \frac{e\mu_B}{\pi d}B(0), \quad (49)$$

i.e., the value $B(0)$ is shifted. Thus, the corrected results are given by replacing the coefficient B_0 in eqs. (26), (27), (31), and (32) as

$$B_0 \to \frac{B_0}{1 + (e\mu\mu_B/\pi d)}. \quad (50)$$

We note that $e\mu\mu_B/\pi d \simeq 5.6 \times 10^{-6}$ for $\mu = \mu_0$ and $d = 10\mathring{A}$.

The result indicates that the helical edge mode shows the weak diamagnetism independent of the parameter g.

6. The Electric Conductivity: Comparision with Superconductivity

Finally, we examine the electric conductivity in our system and compare it with the superconductivity. We integrate out $\vec{a}_0$ from the effective Lagrangian (10). Note that this integration is justified since $\vec{a}_0$ was originally introduced as the auxiliary field of the Stratonovich-Hubbard transformation. We do not integrate out $\vec{\mathbf{a}}$, since this field was introduced merely to estimate the spin current. This integration is straightforward, since the Lagrangian (10) is quadratic with respect to $\vec{a}_0$. We obtain

$$\begin{aligned} \mathcal{L}'_{\mathrm{eff}} &= \sigma^s_{xy}\sum_{ij}\epsilon_{ij}a^z_iE_j - \frac{\sigma^{s2}_{xy}}{2}\boldsymbol{A}^T\frac{\nabla^2}{\epsilon_s\nabla^2 + g^{-1}}\boldsymbol{A}^T\\ &\quad + \frac{\epsilon E^2}{2} - \frac{B^2}{2\mu}\\ &\quad + (\text{terms independent of } a^z_i \text{ and } A_\mu), \quad (51) \end{aligned}$$

where $A^T_i = \sum_j[\delta_{ij} - \partial_i\partial_j/\nabla^2]A_j$ is the transverse (i.e., gauge invariant) component of A_i. We set $\vec{\boldsymbol{a}}$ to be con-

stant after the integration. The quantized SH current is obtained from the first term, as expected. The electric current is[28]

$$\boldsymbol{J} = -\frac{\sigma_{xy}^{s2}\nabla^2}{\epsilon_s\nabla^2 + g^{-1}}\boldsymbol{A}^T. \tag{52}$$

We may take the gauge $A_0 = \boldsymbol{\nabla}\cdot\boldsymbol{A} = 0$ then we have $\boldsymbol{E} = -\dot{\boldsymbol{A}}$ and $\boldsymbol{A}^T = \boldsymbol{A}$. Thus the electric conductivity is

$$\sigma_{xx}(\omega, \boldsymbol{q}) = -\frac{\sigma_{xy}^{s2}\boldsymbol{q}^2}{\epsilon_s\boldsymbol{q}^2 - g^{-1}}\frac{1}{i\omega}, \tag{53}$$

which vanishes in the DC limit taking $\boldsymbol{q} \to 0$ first and $\omega \to 0$ later when $1/g \neq 0$.[12] Namely, the system is insulating. Then, we conclude that the system shows the Meissner effect without infinite DC conductivity when $g > (\sigma_{xy}^{s2}\mu)^{-1}$, in contrast to the superconductivity. A similar result has been obtained by the Maxwell-Chern-Simons (MCS) theory, which is the low-energy and long-wavelength effective theory for the time-reversal-violating topological band insulator, i.e., the quantized Hall system.[29–32]

Eqs. (52) and (53) indicate that the system becomes superconducting at the point $1/g = 0$ [see also refs. 14, 33]. Actually, we can see the infinite DC conductivity from the Kramers-Krönig relation as $\mathrm{Re}[\sigma_{xx}(\omega, 0)] \propto \delta(\omega)$. Unfortunately, this point is difficult to realize in the quantized SH phase, as mentioned in Introduction.

7. Summary

In this study, we investigated the magnetic response in the quantized SH phase of a layered-honeycomb lattice system with the intrinsic spin-orbit coupling λ_{SO} and on-site Hubbard U. When $g \equiv 4Ua^2d/3 < (\sigma_{xy}^{s2}\mu)^{-1}$, where a and d are the lattice constant and interlayer distance, respectively, the magnetic field inside the sample oscillates spatially around a constant value. The oscillation vanishes in the non-interacting limit $U \to 0$. When $g > (\sigma_{xy}^{s2}\mu)^{-1}$, the Meissner effect occurs. It may be possible to see the oscillation or Meissner effect in a superlattice system with an appropriately large a. As a result of Zeeman coupling, the helical edge state[1,2] shows the weak diamagnetism that is independent of g.

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Appendix: Physical implications of BF term

In this Appendix, we summarized the properties of the BF term[7,14–18] in eq. (10)

$$\mathcal{L}_{\mathrm{BF}} = \sigma_{xy}^{s}\epsilon^{\mu\rho\mu}a_\mu\partial_\rho A_\nu, \tag{A·1}$$

where σ_{xy}^{s} is the quantized SHC given by eq. (2).

A.1 Quantized SH effect

The spin current density obtained from the term is

$$j_i^s = \frac{\partial\mathcal{L}_{\mathrm{BF}}}{\partial a_i^z} = \sigma_{xy}^{s}\sum_{j=x,y}\epsilon_{ij}E_j. \tag{A·2}$$

This shows the quantized SH effect.

A.2 Spin accumulation

The spin density is

$$\rho^s = \frac{\partial\mathcal{L}_{\mathrm{BF}}}{\partial a_0^z} = \sigma_{xy}^{s}B, \tag{A·3}$$

where B is the magnetic field perpendicular to the honeycomb lattice layers. It resembles the Zeeman effect, but an essential difference is that the coefficient is not the Bohr magneton but the quantized SHC.

A.3 Dual quantized SH effect

The electric current density is

$$j_i = \frac{\partial\mathcal{L}_{\mathrm{BF}}}{\partial A_i} = \sigma_{xy}^{s}\sum_{j=x,y}\epsilon_{ij}\nabla_j a_0^z. \tag{A·4}$$

This shows that the current flows perpendicular to the gradient of spin chemical potential a_0^z, namely, the spin electric field. This may be called the dual quantized SH effect.

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$$\frac{1}{\mu}\nabla \times \nabla \times \boldsymbol{B} = -\sigma_{xy}^{s2}\frac{\nabla^2}{\epsilon_s \nabla^2 + g^{-1}}\boldsymbol{B}.$$

To solve it, we can obtain the equivalent results discussed in § 4.
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